MODULE 7: RATIONAL NUMBERS, PART 4

Introduction to Decimal Fractions

When we use type setting, writing $\frac{23}{145}$ can cause trouble with our line spacing. So when we write or type we often write 23/145 rather than $\frac{23}{145}$. But this new notation becomes confusing when we use mixed numbers. For example, notice how easy it would be to confuse 31/4 ($\frac{31}{4}$) with 3 1/4 ($3\frac{1}{4}$).

Notice that in 23/145 there is no "top" and "bottom". Numerator and denominator are written on the same line.

The desire to be able to represent rational numbers on the same line coupled with a desire to capitalize on the benefits of place value leads to the invention of *decimal fractions*.

It is easiest to visualize decimal fractions in terms of money. For example we often write \$3.79 but rarely write $\$3\frac{79}{100}$.

Though in writing checks we often write: 3 and 79/100 dollars

The period in \$3.79 is called a *decimal point*. It separates the whole number (dollars) from the fractional part (cents). So let's generalize this idea.

> *Place Value and the Decimal Point*
>
> *A decimal fraction is a place value numeral in which a decimal point appears.*
>
> *The digit* immediately to the left of the decimal point *locates the ones-place.*

Don't confuse decimal fractions with common fractions Decimal fractions have a decimal point. They do not have numerators or denominators.

Example 1

What digit names the ones-place in 23.419?

Answer: 3

All we do is apply the basic definition. Namely, we look at the first digit to the left of the decimal point. Hence we first locate the decimal point:

2 3 . 4 1 9

and we then look for the digit immediately to its left:

2 3 . 4 1 9

While the decimal point works as a place *finder*, 0 still works as a place holder.

That is, the decimal point shows us how to find the ones-place.

Example 2

What place is held by the 3 in 230.419?

Answer: The tens-place

Once we locate the ones-place, we use the same method that we used in Module 1 in describing place value. Namely as we move from right to left we multiply by 10 to get from one place to the next.

The 3 is immediately to the left of the 0, and the 0 holds the ones-place because it is the first digit to the left of the decimal point.

In other words:

hundreds	*tens*	*ones*	. . . . . .
2	3		

The 0 in 230.419 tells us that we have no ones.

So the 3 is in the tens-place.

Note

It is often helpul in studying decimal fractions to think in terms of dollars and cents. For example, when we write \$23.57 it is clear that we have 23 dollars and 57 cents. But when we write \$230.57 we have 230 dollars and 57 cents.

In 23.57 the 3 is in the ones-place but in 230.57 the 3 is in the tens-place.

While we already know how to read the places to the left of the decimal point, we have not yet described how to read the places to the right of the decimal point. The "trick" here is to notice in place value that when we go from left to right we get from one place to the next by *dividing by 10*; and dividing by 10 is the same as multiplying by $\frac{1}{10}$.

In other words, we get from 10 to 100 by multiplying by 10, but to get from 100 to 10 we divide by 10.

So let's see what happens as we go from left to right, starting, say, with 100. $\frac{1}{10}$ of 100 is 10; $\frac{1}{10}$ of 10 is 1; $\frac{1}{10}$ of 1 is $\frac{1}{10}$; $\frac{1}{10}$ of $\frac{1}{10}$ is $\frac{1}{100}$; and so on. In other words, if we start at the decimal point and read from left to right we get:

. *tenths* → *hundredths* → *thousandths* → . . . .

Note:

This is easy to see in terms of money. The first digit to the right of the decimal point is the dimes-place. "Dime" is derived from "decimal". There are 10 dimes per dollars, so each dime is 1/10 of a dollar. The second digit to the right of the decimal point names the cents. place. There are 10 cents per dime, so a cent is 1/10 of a dime or 1/100 of a dollar.

That is, a tenth of a tenth of a dollar is a hundredth of a dollar. "cent" is fro the Latin "centum" which means "hundred".

Example 3

What place does the 4 hold in the decimal fraction 230.419?

Answer: The tenths-place

The 4 is the first digit to the right of the decimal point. Hence it names the tenths place. In more detail:

100	*10*	*1*	*.*	*tenths*	*hundredths*	*thousandths*
2	*3*			*4*	*1*	*9*

In comparing Examples 2 and 3, notice the difference between the tens-place and the tenths-place. Confusing the two is like confusing $10-bills with dimes.

In any event, we can now begin to show the relationship between decimal fractions and common fractions.

Example 4

Write 0.419 as a common fraction in lowest terms.

Answer: $\frac{419}{1,000}$

The decimal point tells us that the number is 4 tenths + 1 hundredth + 9 thousandths. That is:

The 0 in 0.419 is used to emphasize the decimal point. When we write .419 it is possible to overlook the decimal point.

ones	*.*	*tenths*	*hundredths*	*thousandths*
		4	1	9

Hence: $$\frac{4}{10} + \frac{1}{100} + \frac{9}{1,000} = \frac{400}{1,000} + \frac{10}{1,000} + \frac{9}{1,000} = \frac{419}{1,000}$$

The denominator is a power of ten. So the only prime factors of the denominator are 2 and 5. Clearly 419 is divisible by neither 2 nor 5. So 419/1000 is in lowest terms.

There is an interesting relationship between Example 4 and its answer. If we omit the decimal point in .419 the number would be 419. Since the 9 is in the thousandths-place, we have 419 thousandths or $\frac{419}{1,000}$. This agrees with our answer and shows us a shortcut for doing problems like those in Example 4.

Again we seem to return to the common theme that numbers are adjective modifying nouns. In this case 0.419 means 419 thousandths.

Converting Decimal Fractions to Common Fractions:

(1) *Read the decimal fraction as it would appear without the decimal point. This becomes the numerator.*

(2) *Write a 1 followed by as many 0's as there are digits to right of the decimal point. This becomes our denominator.*

(3) *If desired, reduce the common fraction to lowest terms.*

Illustrating this with 0.234 we have:

(1) 234 is the numerator

(2) $0.\underset{1}{2}\underset{2}{3}\underset{3}{4}$ has 3 digits to to the right of the decimal point. So our denominator is 1000

(3) So the fraction is: $\frac{234}{1,000}$ *or* $\frac{117}{500}$

Example 5

Write 0.905 as a common fraction in lowest terms.

Answer: $\frac{181}{200}$

The furthest digit to the right (5) is in the thousandths place. So we have 905 thousandths or $\frac{905}{1,000} = \frac{5 \times 181}{5 \times 200}$

$$= \frac{181}{200}$$

The only prime factors of powers of ten are 2 and 5. Since 181 isn't divisible by either 2 or 5, 181/200 must be in lowest terms.

In terms of our rule, omit the decimal point in 0.905 to get 905 as our numerator. Then since there are 3 digits to the right of the decimal point in 0.905, write 1 followed by three 0's to get 1,000 as our denominator.

We could also have proceeded without the shortcut by writing 0.905 as:

$$\frac{9}{10} + \frac{0}{100} + \frac{5}{1,000} = \frac{9}{10} + \frac{1}{200}$$

$$= \frac{180}{200} + \frac{1}{200}$$

$$= \frac{181}{200}$$

As a review of common fractions, you should be able to supply the details on your own.

Notice that the 0 in .905 is important. Because of it the 5 is in the thousandths place. Without the 0, the 5 is in the hundredths place.

<u>Example 6</u>

Write 0.95 as a common fraction in lowest terms.

Answer: $\frac{19}{20}$

Omitting the decimal point we get 95 as our numerator. Then, since there are 2 digits to the right of the decimal point in 0.95, our denominator is 100. Hence the equivalent common fraction is:

$$\frac{95}{100} = \frac{5 \times 19}{5 \times 20} = \frac{19}{20}$$

In other words, we may read 0.95 as 95 hundredths.

See how Examples 5 and 6 compare? 0.905 is the same as $\frac{181}{200}$ while 0.95 is the same as $\frac{19}{20}$. Since these two common fractions aren't equivalent, neither are the two decimal fractions.

$\frac{19}{20} = \frac{19 \times 10}{20 \times 10} = \frac{190}{200}$

So $\frac{19}{20}$ *exceeds* $\frac{181}{200}$ *by* $\frac{9}{200}$

However, if a 0 doesn't change the place of any non-zero digit, that 0 may be removed or not removed without changing the value of the decimal fraction.

<u>Example 7</u>

Write 0.950 as a common fraction in lowest terms.

Answer: $\frac{19}{20}$

The digit furthest to the right (0) is in the thousandths-place. Hence we have 950 thousandths or $\frac{950}{1,000}$. But:

$$\frac{950}{1,000} = \frac{95 \times 10}{100 \times 10} = \frac{95}{100},$$

and we recognize this as Example 6.

In Example 6 we already found out that:
$\frac{95}{100} = \frac{19}{20}$

If we prefer to use the shortcut, notice that if we omit the decimal point we get 950 as our numerator. And since there are 3 digits to the right of the decimal point, our denominator is a 1 followed by 3 zeros. So we get:

$$\frac{950}{1,000}$$

When we write .950, 0 is one of the digits to the right of the decimal point. That is: $0.\underset{1}{9}\underset{2}{5}\underset{3}{0}$ <u>DON'T FORGET TO COUNT THESE ZEROS</u>.

The point of Example 7 is that when we write 0.950, we're saying that we have no thousandths. We're saying the same thing when we write 0.95.

That is, omitting the 0 in 0.950 doesn't affect the place of either the 9 or 5. But omitting the 0 in .905 changes the 5 thousandths to 5 hundredths.

<u>Note:</u>

As usual, some of these things may be easier to visualize in terms of money. For example, \$9.05 and \$9.50 are different amounts of money. But \$9.5 means 9 dollars plus 5 dimes or \$9.50

That is, \$9.50 says, 9 dollars, 5 dimes, and no cents; which is the same as 9 dollars and 5 dimes. It is a custom to write \$9.50 rather than \$9.5 but it isn't necessary.

Of particular importance are the number of zeros between the decimal point and the first non-zero digit to the right of the decimal point.

<u>Example 8</u>

Write 0.0003 as a common fraction in lowest terms.

Answer: $\frac{3}{10,000}$

If we omit the decimal point in 0.0003 we see that 3 is our numerator. Then since there are 4 digits to the right of the decimal point (that is, $0.\underset{1}{0}\underset{2}{0}\underset{3}{0}\underset{4}{3}$) our denominator is a 1 followed by 4 zeros. Hence our equivalent common fraction is $\frac{3}{10,000}$.

Just as with the odometer of a car we read 00003 as 3

In other words, the 0's in .0003 tell us that we have 3 <u>ten-thousandths</u>. Thus, 3 is the adjective and ten-thousandths is the noun.

So in the language of decimal fractions, 0.0003 tells us that we have 3 parts per 10,000.

Example 9

Write 0.000003 as a common fraction in lowest terms.

Answer: $\frac{3}{1,000,000}$

In this case, the 3 is in the millionths place. That is, the 0's tell us that we're omitting the following places: tenths, hundredths, thousandths, ten-thousandths, and hundred-thousandths.

At this stage we begin to see the advantage of our shortcut. Namely we still get 3 as our numerator but now there are 6 digits to the right of the decimal point. Hence our denominator is a 1 followed by 6 zeros. Thus the equivalent common fraction is: $\frac{3}{1,000,000}$

0.000003
123456

Examples 7 and 8 point out to us how the numerator alone isn't enough to determine the fractional part. Namely 0.0003 names 3 parts per 10,000; while 0.000003 names 3 parts per 1,000,000. More generally:

3 ten-thousandths and 3 millionths are quite different even though 3 is same adjective in both cases.

0.3 means 3 parts per 10
0.03 means 3 parts per 100
0.003 means 3 parts per 1,000
0.0003 means 3 parts per 10,000
0.00003 means 3 parts per 100,000
0.000003 means 3 parts per 1,000,000

See the pattern? The 3 is the same in each case. But the noun is a 1 followed by as many 0's as there are digits to the right of the decimal point.

It might also be interesting to point out that we can think of whole numbers in terms of decimal fractions.

Example 10

What digit names the ones-place in 837 ?

Answer: 7

This is a problem straight from Module 1. The point is that before we introduced decimal fractions, the ones-place was always the digit furthest to the right.

But the ones-place is also the first digit to the left of the decimal point when we use the notation of decimal fractions.

The key point is that when there is no decimal point, we may always assume that it can be placed after the digit furthest to the right.

In other words, just as we can write 12 as 12/1 when we deal with common fractions; we can write 12 as "12." (or 12.0 or 12.00 and so on) when we deal with decimal fractions.

Example 11

In the numeral 2,837 where may we place a decimal point without changing the value of the number?

Answer: After the 7

In 2,837, the 7 is in the ones-place. Since the ones-place is the first digit to the left of the decimal point, we may assume that the decimal point is to the right of the 7. That is:

2, 8 3 7 .

In other words, the decimal point is a place finder. But when we deal with whole numbers, the digit furthest to the right finds the ones-place for us.

Once we realize that decimal fractions behave just like place value whole numbers, the arithmetic of decimal fractions becomes relatively easy. Let's begin by seeing how we add (and subtract) decimal fractions.

Example 12

Find the sum of 2.87 and 5.49 .

Answer: 8.36

Method 1

There are 2 digits to the right of the decimal point in 2.87 so our denominator would be a 1 followed by 2 zeros or 100.

Without the decimal point the number is 287. Hence as a common fraction 2.87 may be written as $\frac{287}{100}$. In a similar way we see that 5.49 may be written as $\frac{549}{100}$.

Therefore, $2.87 + 5.49 = \frac{287}{100} + \frac{549}{100}$

$$= \frac{287 + 549}{100}$$

$$= \frac{836}{100}$$

What we're doing here is to translate the problem into the language of common fractions, solve it in that language, and then translate the answer back into the language of decimal fractions

This is the same as 836 hundredths. To put the 6 in the hundredths-place we write the answer in decimal form as 8.36

Method 2 (A paraphrase of Method 1)

In terms of whole number adjectives, 2.87 means 287 *hundredths* and 5.49 means 549 *hundredths*. Hence 2.87 + 5.49 means:

```
  287 hundredths
+ 549 hundredths
  836 hundredths
```

This emphasizes the fact that we may view decimal fraction addition the same as we viewed whole number addition. We added the two whole numbers (287 and 549) and kept the common denomination (hundredths).

Method 3

Line the numbers up vertically so that the decimal points are in the same column:

```
  2 . 8 7
+ 5 . 4 9
```

Place the decimal point of the answer directly under the other two decimal points:

```
  2 . 8 7
+ 5 . 4 9
    .
```

Then add the same as if the decimal points were not there:

$$\begin{array}{r} 2\,.\,8\;7 \\ +\;5\,.\,4\;9 \\ \hline 8\,.\,3\;6 \end{array}$$

This is a different form of Method 2. The only difference is that now we're letting the decimal point take the place of the word "hundredths"

Note:

It's important to use the decimal point to find the like denominations. For example, if we we were to write:

$$\begin{array}{r} 2.4\;6 \\ \underline{4\;1\;5.7} \end{array}$$

we couldn't say that 6 and 7 were 13 because we'd be adding 6 hundredths and 7 tenths. To use the addition tables we must add common denominations.

Think in terms of money again. We can add 7 cents and 9 cents to get 16 cents; we may add 8 dimes and 4 dimes to get 12 dimes; and we may add 2 dollars and 5 dollars to get 7 dollars:

	dollars	*dimes*	*cents*
	2	*8*	*7*
+	*5*	*4*	*9*
	7	*12*	*16*
	7	*13*	*6*
	8	*3*	*6*

but if we add 7 cents and 4 dimes we get neither 11 cents nor 11 dimes.

The usual procedure for adding decimal fractions is to use Method 3.

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To Add Two or More Decimal Fractions:

(1) Write the fractions in vertical form so that the decimal points are in the same column.

(2) In the answer, write the decimal point directly below the other decimal points.

(3) Then add as if we were dealing with whole numbers. That is, pretend there is no decimal point.

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Example 13

Write 2.34 + 15.2 + 6.517 as a decimal fraction.

Answer: 24.057

The most direct way is to use the decimal points as place finders. In this context we have:

tens	*ones* .	*tenths*	*hundredths*	*thousandths*	
	2	3	4		
1	5	2			
	6	5	1	7	
1	13	10	5	7	=
1	14	(0)	5	7	=
2	4	(0)	5	7	

and if we now omit the denominations and use the decimal point we get 24.057

Method 2 (The "Recipe")

Line the numbers up vertically so that the decimal points are in the same column:

```
  2 . 3 4
1 5 . 2
  6 . 5 1 7
```

and we then add as if there were no decimal points. That is:

```
  2 . 3 4 0
1 5 . 2 0 0
  6 . 5 1 7
-----------
2 4 . 0 5 7
```

We may rewrite 2.34 as 2.340 and 15.2 as 15.200. This allows us to rewrite the problem as:

```
 2.340
15.200
 6.517
```

and in this way, it seems easier to keep the common denominations in line.

But be careful in where you place the 0's. 15.002 is not the same as 15.2 nor is 2.034 the same as 2.34

In terms of adjectives and nouns we have:

```
 2,340 thousandths
15,200 thousandths
 6,517 thousandths
------------------
24,057 thousandths
```

In this form we are emphasizing the role of whole number arithmetic.

Subtracting decimal fractions follows in almost exactly the same fashion.

Example 14

Express 5.32 - 2.87 as a decimal fraction.

Answer: 2.45

Method 1 (The Place Finder Technique)

	ones .	*tenths*	*hundredths*	
	5	*3*	*2*	=
	5	*2*	*12*	=
	4	*12*	*12*	
-	*2*	*8*	*7*	
	2	*4*	*5*	

Notice how this is exactly the same as whole number place value arithmetic.

245 hundredths = 2.45 That is, the 5 must be in the hundredths-place.

Method 2 (Lining Up the Decimal Points)

We line up the decimal points, just as we did for addition. We get:

$$\begin{array}{r} 5\,.\,3\;2 \\ -\;\underline{2\,.\,8\;7} \\ . \end{array}$$

Then we subtract as if there were no decimal points:

$$\begin{array}{r} 4\;\;\;{}^{1}2\;\;\;\; \\ \cancel{5}\,.\,\cancel{3}\;{}^{1}2 \\ -\;\underline{2\,.\,8\;\;7} \\ 2\,.\,4\;\;5 \end{array}$$

In terms of whole numbers, we have:

532 hundredths
- 287 hundredths
245 hundredths

Method 3 (Converting to Common Fractions)

$$5.32 = \frac{532}{100} \text{ and } 2.87 = \frac{287}{100}$$

Therefore:

$$\begin{aligned} 5.32 - 2.87 &= \frac{532}{100} - \frac{287}{100} \\ &= \frac{532 - 287}{100} \\ &= \frac{245}{100} \\ &= 245 \text{ hundredths} \\ &= 2.45 \end{aligned}$$

In the same way that 245,000 takes the place of writing 245 thousand, 2.45 takes the place of writing 245 hundredths.

Remember that the meaning of subtraction is the same as it was in the preceeding modules.

Example 15

What must we add to 2.87 to get 5.32 as the sum?

Answer: 2.45

This is a rewording of Example 14. In terms of fill-in-the-blank, we have:

2.87 + ______ = 5.32

and this is the same as _____ = 5.32 - 2.87.

So we have a subtraction problem that looks like an addition problem. Be careful we aren't adding 2.87 and 5.32

As a check we have:

$$\begin{array}{r} 2.45 \\ +\;\underline{2.87} \\ 5.32 \end{array}$$

In subtraction we have to be a bit more careful than with addition.

Example 16

Express 5.32 - 2.871 as a decimal fraction.

Answer: 2.449

In this case we have:

	ones	.	*tenths*	*hundredths*	*thousandths*
	5		3	2	
-	2		8	7	1

We can't take 1 thousandth from no thousandths so we have to "borrow". That is, we rewrite the problem in the form:

	ones	.	*tenths*	*hundredths*	*thousandths*	
	5		3	2	0	=
	5		3	1	10	=
	5		2	11	10	=
	4		12	11	10	
-	2		8	7	1	
	2		4	4	9	

The key point here is DON'T write $\begin{array}{r} 5.32 \\ -2.871 \\ \hline 1 \end{array}$

The 1 is being subtracted from 0 (or 10) and this is going to result in a 9 not a 1 in the thousandths-place.

and this is the same as 2.449

In terms of whole numbers, we have:

5.32 = 5.320 = 5,320 thousandths
2.871 = 2,871 thousandths

Therefore the problem is the same as:

$$\begin{array}{rl} & 5{,}320 \text{ thousandths} \\ - & 2{,}871 \text{ thousandths} \\ \hline & 2{,}449 \text{ thousandths or } 2.449 \end{array}$$

We'll end this Module with a discussion of how we multiply decimal fractions. The problem of dividing decimal fractions will be left for the next module.

We'll begin our discussion of multiplication by restating in terms of decimal fractions how we multiply and divide by 10.

Example 17

At \$2.34 each, how much will 10 items cost?

Answer: \$23.40

We want the sum of ten \$2.34's. If we prefer to work in terms of whole numbers, we want the sum of ten 234's or 234 X 10.

That is, \$2.34 = 234¢

From Module 3 we know that 234 X 10 = 2,340. Hence the total cost is 2,340 cents. In terms of dollars we write this as \$23.40

We could have written \$23.4 but in writing money amounts, we hold both the dimes and cents places.

If you compare the problem with the answer in Example 17, notice that, in effect, to multiply 2.34 by 10 we simply moved the decimal point one place to the *right*.

Conversely, if we knew that the total price of the ten items was \$23.40, we'd have moved the decimal point one place to the *left* to find the price of each item.

This generalizes what we said in Module 3. For example when we multiply 24 by 10 to get 240, the decimal point was originally after the 4 (that is, 24.) We then used 0 as a place holder when we moved the decimal point one place to the right. That is:

24.0.

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Multiplying and Dividing Decimal Fractions By Ten

To multiply a decimal fraction by 10 move the decimal point one place to the right.

To divide a decimal fraction by 10 move the decimal point one place to the left.

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If there is no decimal point we may place one after the digit furthest to the right without changing the value of the number.

Be careful about saying to add a 0. For example if we add a 0 to 4.3 we get 4.30 which is the same as 4.3--not ten times 4.3. To get 43 we have to move the decimal point one place to the right.

Once we know how to multiply and divide by 10 with decimal fractions, we can multiply and divide by any power of 10.

Example 18

Write 0.02345 X 1,000 as a decimal fraction.

Answer: 23.45

We may think of 1,000 as being the product of three 10's. That is;

$1,000 = 10 \times 10 \times 10 = 10^3$

So:

0.02345 X 1,000 =

0.02345 X (10 X 10 X 10) =

(0.02345 X 10) X (10 X 10) =

(0.2345) X (10 X 10) =

(0.2345 X 10) X 10 =

(2.345) X 10 =

23.45

We're using the associative property of multiplication

Moving the decimal point 1 place to the right converts 0.02345 into 00.2345 which is the same as .2345

To get from 0.02345 to 23.45 we moved the decimal point 3 places to the right. That is, we multiplied by 10 three times, and each time we multiplied by 10 we moved the decimal point one place to the right. More generally:

To Multiply or Divide Decimal Fractions By Powers of Ten:

To multiply a decimal fraction by 10^n we move the decimal point n places to the right.

To divide a decimal fraction by 10^n we move the decimal point n places to the left.

Recall that 10^n is the product of n ten's. For example $10^4 = 10 \times 10 \times 10 \times 10 = 10,000$

Example 19

Write 23,456.7 ÷ 10,000 as a decimal fraction.

Answer: 2.34567

10,000 is 10^4. According to the rule to divide by 10^4 we move the decimal point 4 places to the left. This gives us:

Check: $2.34567 \times 10^4 = 2\underbrace{3456}_{1234}.7$

2.3 4 5 6 ‸ 7
4 3 2 1

Rough Check

20,000 ÷ 10,000 = 2, so 23,456.7 ÷ 10,000 should be a "little" more than 2.

2.34567 is a "little" more than 2.

As a closer check 23,456 ÷ 10,000 =

$$\begin{array}{r} 2 \\ 10{,}000\overline{)23{,}456} \\ \underline{-20{,}000} \\ 3{,}456 \end{array}$$

or $2\frac{3,456}{10,000}$ and as a decimal fraction this is 2.3456 which is very close to 2.34567

Remember that .3456 is 3456 "over" a 1 followed by 4 zeros.

In fact, our rough check actually gives us an exact way to convert decimal fraction problems into common fraction or mixed number problems. Recall that without the decimal point 23,456.7 would look like 234,567 and since the 7 is in the tenths place we know that 23,456.7 is 234,567 tenths or $\frac{234,567}{10}$.

This is a very powerful result. It says that we can solve new problems by replacing them by equivalent easier problems. By "easier we mean problems that we can already solve.

Therefore:

$$\begin{aligned} 23{,}456.7 \div 10{,}000 &= \frac{234{,}567}{10} \div \frac{10{,}000}{1} \\ &= \frac{234{,}567}{10} \times \frac{1}{10{,}000} \\ &= \frac{234{,}567}{100{,}000} \\ &= 2\frac{34{,}567}{100{,}000} \\ &= 2.34567 \end{aligned}$$

.34567 is another way of saying 34,567 hundred-thousandths.

Let's apply this approach to the problem of multiplying any two decimal fractions.

Example 20

Write 0.07 X 0.003 as a decimal fraction.

Answer: 0.00021

If we use the language of common fractions, 0.07 means the same as $\frac{7}{100}$ and 0.003 means the same as $\frac{3}{1,000}$. Therefore:

$$0.07 \times 0.003 = \frac{7}{100} \times \frac{3}{1,000}$$

$$= \frac{7 \times 3}{100 \times 1,000}$$

$$= \frac{21}{100,000}$$

= 21 hundred-thousandths

= 0.00021

So we start with 21. and move the decimal point 5 places to the left:

$0\,.\,\underset{5}{0}\,\underset{4}{0}\,\underset{3}{0}\,\underset{2}{2}\,\underset{1}{1}\,.$

Notice the steps we actually used in doing this example.

Step 1: *Pretend that there are no decimal points. In this way, 0.07 becomes 007 which is the same as 7 and 0.003 becomes 0003 which is the same as 3.*

These are precisely the numerators we get when we translate the decimal fractions into common fractions.

Step 2: *Multiply the two numbers that you got in Step 1; that is, multiply 7 by 3.*

This corresponds to multiplying the numerators of the common fractions.

Step 3: *Count the number of digits that are to the right of the decimal points in the original two decimal fractions. In this case:*

$0.\,\underset{1}{0}\,\underset{2}{7} \qquad 0.\,\underset{3}{0}\,\underset{4}{0}\,\underset{5}{3}$

This gives us the total number of 0's in the denominators of the common fractions.

Step 4: *Now take the answer you got in Step 2 (21.) and move the decimal place as many places to the left as there were digits in Step 3 (that is, 5 places to the left).*

The result is the answer; namely: $0.\underset{5}{0}\,\underset{4}{0}\,\underset{3}{0}\,\underset{2}{2}\,\underset{1}{1}$

Each digit represents a 10 that we divided by and since there were five factors of 10, we move the decimal point a total of 5 places to the left.

See how the 0's act as place holders? We had only 2 digits to the left of the decimal point, so we had to annex three 0's to move the decimal point 5 place to the left.

These four steps tell us how to multiply any two decimal fractions.

Example 21

Write 3.14 X 2.7 as a decimal fraction.

Answer: 8.478

Let's use the four steps.

(1) Omit the decimal points to get 314 and 27.

(2) Multiply the two numbers you got in (1).

$$\begin{array}{r} 314 \\ \times\ 27 \\ \hline 2198 \\ +\ 628 \\ \hline 8478. \end{array}$$

(3) Count the digits to the right of the decimal points:

$$3.\underset{1}{1}\underset{2}{4} \qquad 2.\underset{3}{7}$$

(4) Since there are 3 digits to the right of the decimal points, move the decimal point in Step (2) 3 places to the left to get 8.478

Note:

If you get mixed up in placing the decimal point, there is a fairly reasonable way to guess where to put it. For example, 3.14 rounds off to 3 and 2.7 rounds off to 3. Hence by rounding off, 3.14 X 2.7 is approxiametely 3 X 3 or 9. In 8478 the only way we can place the decimal point to get a number close to 9 is between the 8 and 4. That is, we feel the answer must be 8.478

In fact:

3.14 is between 3 and 4

2.7 is between 2 and 3

So the product is between $\begin{array}{r} 3 \\ \times\ 2 \\ \hline 6 \end{array}$ *and* $\begin{array}{r} 4 \\ \times\ 3 \\ \hline 12 \end{array}$

As you study the steps, notice what happens. You multiply:

$$\begin{array}{r} 3.14 \\ \times\ \ 2.7 \\ \hline \end{array}$$

ignoring the decimal points (The decimal points do not have to be in the same column) to get:

$$\begin{array}{r} 3.14 \\ \times\ 2.7 \\ \hline 2198 \\ 628 \\ \hline 8478. \end{array}$$

and then move the decimal point 3 places to the left to get:

$$8.\underbrace{478}$$

In common fraction form, what we did was:

$$3.14 \times 2.7 = \frac{314}{100} \times \frac{27}{10}$$

$$= \frac{314 \times 27}{100 \times 10}$$

$$= \frac{8,478}{1,000}$$

$$= 8.478$$

And to get an answer that's between 6 and 12, we have to write 8.478.

In closing this module, we'd be remiss if we did not mention a few words about computers and calculators. In many respects common fractions are easier to understand than are decimal fractions. But the fact that decimal fractions are so closely related to the place value of whole number arithmetic makes them very important in any system that uses place value principles. In particular, calculators are designed this way.

If you want to divide 3 by 16 on a calculator the answer does not appear in the form $\frac{3}{16}$. Rather it will appear as the equivalent decimal fraction--which in this case happens to be 0.1875. How we get this result without the calculator will be part of the discussion in the next module.

It's probably easier to think in terms of 3 out of each 16 rather than in terms of 1,875 out of each 10,000; but we must use the latter if we're using decimal fractions.

But the main point is that in our modern, highly-technological society, decimal fractions are encountered with more frequency than any other type.

In any event, since each type of fraction has definite advantages (as well as disadvantages), it is extremely helpful to be able to translate from any form to any other form.